A local Langevin equation for slow long-distance modes of hot non-Abelian gauge fields

Dietrich Bödeker [1]

Physics Department, Brookhaven National Laboratory and
RIKEN BNL Research Center, Upton, NY 11973

Abstract

The effective theory for the dynamics of hot non-Abelian gauge fields with spatial momenta of order of the magnetic screening scale g^2T is described by a Boltzmann equation. The dynamical content of this theory is explored. There are three relevant frequency scales, gT , g^2T and g^4T, associated with plasmon oscillations, multipole fluctuations of the charged particle distribution, and with the non-perturbative gauge field dynamics, respectively. The frequency scale gT is integrated out. The result is a local Langevin-type equation. It is valid to leading order in g and to all orders in $\log(1/g)$, and it does not suffer from the hard thermal loop divergences of classical thermal Yang-Mills theory. We then derive the corresponding Fokker-Planck equation, which is shown to generate an equilibrium distribution corresponding to 3-dimensional Yang-Mills theory plus a Gaussian free field.

PACS numbers: 11.10.Wx, 11.15.-q, 95.30.Cq

[1] e-mail: bodeker@bnl.gov

At sufficiently high temperature T the running gauge coupling $g = g(T)$ in non-abelian gauge theories is small. Nevertheless, long distance modes of hot non-abelian gauge fields are strongly coupled. This leads to the phenomenon of magnetic screening on the length scale $(g^2T)^{-1}$. [2] It distinguishes non-abelian from abelian plasmas in which magnetic fields can be correlated over arbitrarily large distances.

The strong coupling is caused by the large amplitudes of the infrared gauge field modes. It can be easily understood as follows. The long wavelength ($\lambda \gg T^{-1}$) modes obey the classical Rayleigh-Jeans law which states that the energy density is proportional to T. Then, by dimensional analysis, the magnetic energy density $\mathbf{B}_\lambda^2$ due to wavelengths of order λ must be of order $T\lambda^{-3}$. This corresponds to $\mathbf{B}_\lambda \sim T^{1/2}\lambda^{-3/2}$ and to a vector potential $\mathbf{A}_\lambda \sim T^{1/2}\lambda^{-1/2}$. The Yang-Mills equations of motion are non-linear since they contain covariant derivatives. When λ approaches $(g^2T)^{-1}$, both terms in the spatial covariant derivative $\mathbf{D} = \nabla - g\mathbf{A}$ become of order g^2T, and the term $g\mathbf{A}$ can no longer be treated as a perturbation.

This breakdown of perturbation theory was first noted in the thermodynamics of hot non-abelian gauge fields [1]. The non-perturbative physics can be ascribed to euclidean pure Yang-Mills theory in 3 dimensions which is obtained from the 4-dimensional thermal field theory in the imaginary time formalism by integrating out all modes with characteristic wavelengths shorter than $(g^2T)^{-1}$ [2]. The 3-dimensional theory can be easily treated non-perturbatively on a lattice.

Dynamical quantities, which are determined by the real (Minkowski) time evolution of the $\lambda \sim (g^2T)^{-1}$ modes, are more difficult to deal with. An important example is the so called hot sphaleron rate. In the standard electroweak theory it determines the rate for anomalous baryon number violation and is therefore a crucial ingredient in particle physics models which try to explain the baryon asymmetry of the universe [3].

Fortunately, even for dynamical quantities one can use perturbation theory to integrate out short distance modes to obtain an effective theory for the non-perturbative long-distance dynamics. The first step is to integrate out "hard" physics associated with virtual momenta of order T. At leading order in g one obtains the so called hard thermal loop effective theory [4, 5]. It describes gauge fields with $\lambda \gtrsim (gT)^{-1}$ interacting with classical colored particles. These particles correspond to quanta of the $|\mathbf{k}| \sim T$ field modes which have virtual momenta of order gT or less. In a second step one can integrate out the degrees of freedom associated with $k_0, |\mathbf{k}| \sim gT$. The resulting effective theory is described by the classical field equations of motion [6]

$$\begin{aligned} D_\mu F^{\mu\nu} &= m_{\mathrm{D}}^2 W^\nu, && \text{(1.a)} \\ (C + v \cdot D)W &= \mathbf{v} \cdot \mathbf{E} + \xi, && \text{(1.b)} \end{aligned}$$

where $F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{abc} A_b^\mu A_c^\nu$ is the non-abelian field strength tensor, and $v^\mu \equiv (1, \mathbf{v})$. The field $W(x, \mathbf{v})$ represents the fluctuations of adjoint color charge due to hard particles with 3-velocity $\mathbf{v}$, $\mathbf{v}^2 = 1$ [3]. For other approaches leading to the

[2]The units are chosen such that $\hbar = c = k_{\mathrm{B}} = 1$. Furthermore, it is assumed that fermion masses and chemical potentials can be neglected relative to T.

[3]For notational simplicity we write all expressions in 3 spatial dimensions. One has to keep in

Boltzmann equation (1.b) which do not make use of the hard thermal loop effective theory, see [7]-[10]. The current on the rhs of Eq. (1.a) is given by

$$W^{\nu}(x) \equiv \int_{\mathbf{v}} v^{\nu} W(x, \mathbf{v}), \tag{2}$$

where $\int_{\mathbf{v}} \equiv \int d\Omega_{\mathbf{v}}/(4\pi)$, times the square of the leading order Debye mass $m_{\rm D} \propto gT$. The lhs of Eq. (1.b) contains the linear collision term

$$CW(x, \mathbf{v}) \equiv \int_{\mathbf{v}'} C(\mathbf{v}, \mathbf{v}') W(x, \mathbf{v}'), \tag{3}$$

which is due to scattering of hard particles with velocities $\mathbf{v}$ and $\mathbf{v}'$ with a momentum transfer of order gT. The fields in Eq. (1) describe fluctuations close to thermal equilibrium on length scales larger than $(gT)^{-1}$. The collision term, breaking time reflection invariance, is dissipative. It is accompanied by the Gaussian white noise ξ, which has vanishing expectation value. Its only non-trivial correlation function is

$$\langle \xi_a(x, \mathbf{v}) \xi_b(x', \mathbf{v}') \rangle = \frac{2T}{m_{\rm D}^2} C(\mathbf{v}, \mathbf{v}') \delta_{ab} \delta^4(x - x'). \tag{4}$$

The collision kernel $C(\mathbf{v}, \mathbf{v}')$ is of order $g^2 T$. Its precise form is not important for the purpose of this letter. We will only use the fact that C commutes with rotations in $\mathbf{v}$-space. Thus if $W(x, \mathbf{v})$ is expanded in spherical harmonics, the collision operator becomes diagonal and its eigenvalues c_l only depend on l, i.e.,

$$\int_{\mathbf{v}'} C(\mathbf{v}, \mathbf{v}') \sum_{lm} W_{lm}(x) Y_{lm}(\mathbf{v}') = \sum_{lm} c_l W_{lm}(x) Y_{lm}(\mathbf{v}). \tag{5}$$

C consists of a piece which depends logarithmically on the infrared cutoff for the $|\mathbf{k}| \sim gT$ modes and of a cutoff independent part. The $l = 0$ eigenvalue c_0 vanishes. For the logarithmic part this was shown in Ref. [6]. In Ref. [11] it was found that this holds for the complete c_0 if one uses dimensional regularization to define C. The vanishing of c_0 ensures that Eq. (1.b) is consistent with Eq. (1.a), i.e., that the current on the rhs of Eq. (1.a) is conserved. The complete $l = 1$ eigenvalue c_1 in dimensional regularization was explicitly calculated in Ref. [12].

At leading order in $\log(1/g)^{-1}$ Eq. (1) can be approximated by the Langevin equation [6]

$$\mathbf{D} \times \mathbf{B} = \gamma \mathbf{E} + \boldsymbol{\zeta}, \tag{6}$$

where $E^i = F^{i0}$ and $B^i = -\frac{1}{2}\epsilon^{ijk} F^{jk}$ are the non-abelian electric and magnetic fields. The damping coefficient or color conductivity γ is proportional to $T/\log(1/g)$. The Gaussian white noise $\boldsymbol{\zeta}$ satisfies

$$\langle \zeta_a^i(x) \zeta_b^j(x') \rangle = 2\gamma T \delta_{ab} \delta^{ij} \delta(x - x'). \tag{7}$$

mind that the effective theories discussed here, except Eq. (6), require regularization, for example by a continuation to $d = 3 - 2\epsilon$ dimensions.

Therefore it keeps the gauge fields in thermal equilibrium at temperature T. Eq. (6) implies that the characteristic frequency of the gauge fields is of order $\log(1/g)g^4T$ [13].

One motivation for integrating out the momentum scale $|\mathbf{k}| \sim gT$ was that the Hard Thermal Loop effective theory contains UV divergences which cannot be removed by renormalization. Thus simulations of the non-perturbative gauge field dynamics using the HTL effective theory do not have a continuum limit. This problem still persists in (1), however. If one would try to take the continuum limit of (1) by sending the UV cutoff to infinity one would encounter precisely the same divergences, simply because the collision term and the noise, which distinguish Eq. (1) from the HTL effective theory, can be neglected at very large momenta as they do not grow as fast as the derivative terms.

The effective theory (6), on the other hand, is UV finite [7]. Thus it is well suited for non-perturbative lattice simulations. It was used to compute the hot sphaleron rate by Moore [14]. In Ref. [15] it was extended to account for a Higgs field with a thermal mass of order g^2T.

Recently Arnold obtained a non-local Langevin equation for the $k_0 \sim g^4T$ dynamics which is valid to leading order in g and to all orders in $\log(1/g)^{-1}$ [16]. Arnold and Yaffe showed that it can be used to systematically improve the theory (6) in a perturbative expansion in $\log(1/g)^{-1}$ [12]. They found that Eq. (6) is still valid at next-to-leading order in $\log(1/g)^{-1}$ if one includes a next-to-leading log correction in the color conductivity γ [12]. With this correction Moore's result for the hot sphaleron rate [14] agrees surprisingly well with different simulations of the hard thermal loop effective theory [17]-[19], which include all orders in $\log(1/g)^{-1}$ but which do not have a continuum limit [20].

The purpose of this letter is to fully explore the dynamical content of Eq. (1). It will be shown that, in addition to the well known plasmon oscillations and the non-perturbative gauge field dynamics, Eq. (1) describes fluctuations of multipole moments of W with a characteristic frequency of order g^2T. Then we will obtain a generalization of Eq. (6) by integrating out the physics of plasmon oscillation which is characterized by the frequency scale gT. The result, Eq. (16), is a Langevin equation which is local in space and time. In contrast to Eq. (6) and to the Langevin equation of Ref. [16] it contains two different frequency scales, g^2T and g^4T.

In the following power counting estimates logarithms of g will be ignored. All approximations will be valid at leading order in g and all orders in $\log(1/g)^{-1}$. Sometimes it will be convenient to write the scalar, vector, and multipole components of W separately. The latter will be denoted by $\widetilde{W}$, so that

$$W(\mathbf{v}) = W^0 + 3\mathbf{v}\cdot\mathbf{W} + \widetilde{W}(\mathbf{v}), \tag{8}$$

where the factor 3 simply follows from the definition (2) and from $\int_{\mathbf{v}} v^i v^j = \frac{1}{3}\delta^{ij}$.

The only length scale in Eq. (1) is set by C^{-1} and the magnetic screening length which are both of order $(g^2T)^{-1}$. We have already seen that the covariant derivative $\mathbf{D}$ is then of the same order of magnitude as the ordinary derivative. Therefore one can obtain important information about the dynamical content of Eq. (1) already by

considering the linearized equations of motion. We Fourier transform them,

$$\begin{aligned} ik^0\mathbf{E} + i\mathbf{k}\times\mathbf{B} &= m_\mathrm{D}^2\mathbf{W}, & (9.\mathrm{a})\\ i\mathbf{k}\cdot\mathbf{E} &= m_\mathrm{D}^2 W^0, & (9.\mathrm{b})\\ (C - iv\cdot k)W &= \mathbf{v}\cdot\mathbf{E} + \xi, & (9.\mathrm{c}) \end{aligned}$$

and consider only $|\mathbf{k}| \sim g^2T$. The magnetic field can be eliminated from Eq. (9.a) using $\mathbf{k}\times\mathbf{E} = k^0\mathbf{B}$,

$$ik_0\mathbf{E} + \frac{i}{k^0}\mathbf{k}\times\mathbf{k}\times\mathbf{E} = m_\mathrm{D}^2\mathbf{W}. \tag{10}$$

First consider the case $k_0 \gg |\mathbf{k}|$. Then one can neglect the second term on the lhs of Eq. (10), so that

$$\mathbf{E} = -i\frac{m_\mathrm{D}^2}{k^0}\mathbf{W}. \tag{11}$$

Combining Eqs. (9.b) and (11) one finds that $W^0 \sim \mathbf{k}\cdot\mathbf{W}/k^0 \ll |\mathbf{W}|$. Thus one can neglect W^0 in Eq. (9.c). Since $k_0 \gg C \sim g^2T$ one can neglect the collision term and the noise in Eq. (9.c) and one can approximate $v\cdot k \simeq k_0$, which gives

$$k_0\left(3\mathbf{v}\cdot\mathbf{W} + \widetilde{W}\right) = \frac{m_\mathrm{D}^2}{k^0}\mathbf{v}\cdot\mathbf{W}. \tag{12}$$

If one multiplies Eq. (12) with $\mathbf{v}$ and integrates over $\mathbf{v}$, $\widetilde{W}$ drops out and one obtains

$$\left(k_0^2 - \frac{1}{3}m_\mathrm{D}^2\right)\mathbf{W} = 0. \tag{13}$$

Thus $\mathbf{W}$ oscillates with the plasmon frequency $\omega_\mathrm{pl} = \frac{1}{\sqrt{3}}m_\mathrm{D}$. The plasmon oscillations also involve the electric field which is determined by Eq. (11), and W^0 which is obtained from $\mathbf{E}$ using Eq. (9.b). For $\widetilde{W}$, Eq. (12) would imply $\widetilde{W} \propto \delta(k^0)$, meaning that $\widetilde{W}$ is time independent. But Eq. (12) is only an approximation valid when $k^0 \gg g^2T$. Thus from Eq. (12) one can only conclude that $\widetilde{W}$ evolves more slowly than W^μ. We will now see that the characteristic frequency of $\widetilde{W}$ is of order g^2T.

The appearance of the frequency scale g^2T, which so far has not been discussed in the literature, is immediately obvious if we choose the z-axis in Eq. (9) parallel to $\mathbf{k}$ and expand $W(\mathbf{v})$ in spherical harmonics $Y_{lm}(\mathbf{v})$. Then the factor $v\cdot k$ in (9.c) equals $k^0 - |\mathbf{k}|\cos\theta_\mathbf{v}$ and does not mix W_{lm} with different m. Furthermore, the collision term is diagonal (cf. Eq. (5)), so that the W_{lm} with $|m| \geq 2$ are completely decoupled from the gauge fields. The dynamics of these modes is thus governed by Eq. (9.c) but without the electric field. They perform oscillations with frequencies determined by $\mathbf{v}\cdot\mathbf{k} \sim g^2T$, which are driven by the noise term ξ, and which are damped by the collision term at a rate of order $C \sim g^2T$.

To include the $|m| \leq 1$ modes of W and the gauge fields in this picture it is convenient to solve Eq. (9.c) for W,

$$W(\mathbf{v}) = \int_{\mathbf{v}'} G_k(\mathbf{v},\mathbf{v}')\Big[\mathbf{v}'\cdot\mathbf{E} + \xi(\mathbf{v}')\Big]. \tag{14}$$

Here $G_k(\mathbf{v},\mathbf{v}')$ denotes the $\mathbf{v}$-space inverse of the operator $C - iv\cdot k$. It has cuts in the lower half of the complex k^0-plane which reflects the damping caused by the collision term. To understand the role of the electric field in Eq. (14) we insert this result into Eq. (10),

$$ik_0\mathbf{E} + \frac{i}{k^0}\mathbf{k}\times\mathbf{k}\times\mathbf{E} = m_\mathrm{D}^2\int_{\mathbf{v},\mathbf{v}'}\mathbf{v}G_k(\mathbf{v},\mathbf{v}')\Big[\mathbf{v}'\cdot\mathbf{E} + \xi(\mathbf{v}')\Big]. \tag{15}$$

Since we consider $k_0 \sim |\mathbf{k}| \sim g^2T$, both terms on the lhs of Eq. (15) are of order $g^2T\mathbf{E}$, while $m_\mathrm{D}^2 G_k\mathbf{E}$ on the rhs is of order $T\mathbf{E}$. Consequently the lhs of Eq. (15) can be neglected completely when $k_0 \sim g^2T$. This is a crucial point which will later allow us to drop the term $D_0\mathbf{E}$ for all frequencies smaller than gT, even though both terms on the lhs of Eq. (15) are of the same order of magnitude when $k^0 \sim g^2T$. It implies that the two terms on the rhs of Eq. (15) must cancel. The electric field is thus entirely determined by the noise ξ in such a way that the current $m_\mathrm{D}^2\mathbf{W}$ vanishes at leading order in g. Gauss' law (9.b) , together with Eq. (14) gives $W^0 \sim g^2\widetilde{W}$. Therefore both W^0 and $\mathbf{W}$ are small compared to $\widetilde{W}$ when $k_0 \sim g^2T$.

Finally we briefly recall the case $k_0 \ll g^2T$ which has been studied in great detail [6], [21]-[23], [16]. Again, Gauss' law implies that one can neglect W^0 in Eq. (9.c). Furthermore, one can approximate $v\cdot k \simeq -\mathbf{v}\cdot\mathbf{k}$. This corresponds to dropping D_0W in Eq. (1.b). Then W is not dynamical, but it is fixed by the gauge fields and the noise at the same instant of time. For $k_0 \ll g^2T$ the "magnetic" term $k_0^{-1}\mathbf{k}\times\mathbf{k}\times\mathbf{E}$ in Eq. (15), which so far has not played any role, becomes relevant. It is now much larger than the kinetic term $k_0\mathbf{E}$ which can be neglected. Then Eq. (15) gives $k_0 \sim g^4T$ as the characteristic frequency of the magnetic sector.

To summarize the discussion of the linearized equations of motion (9), we have found that the characteristic frequency of the electric field $\mathbf{E}$ and the 4-current W^μ is given by $\omega_\mathrm{pl} \sim gT$. The characteristic frequency of $\widetilde{W}$, i.e., of the $l \geq 2$ components of W is $k^0 \sim g^2T$. Finally, the characteristic frequency of the magnetic fields is of order g^4T.

Now consider the effect of interactions in Eq. (1). Since $\mathbf{D} \sim \nabla$ it is clear that none of the above order of magnitude estimates is changed by replacing $\nabla \to \mathbf{D}$. The basic picture of plasmon oscillations and multipole oscillations is unaffected, except that they occur in a quasi-static gauge field background.

The effect of replacing ∂_0 by D_0 is less obvious and it depends on which frequencies are involved. First consider the plasmon oscillations. One can estimate the size of A^0 using $\nabla A^0 \sim \mathbf{E}$. Because of equipartition $\mathbf{E}(x)$ and $\mathbf{B}(x)$ are of the same order of magnitude. This gives the same estimate for $A^0(x)$ as for $\mathbf{A}(x)$, i.e., $A^0(x) \sim gT$. Consequently one can neglect gA^0 in the covariant time derivative acting on fields with frequencies of order gT, and Eq. (13) and the corresponding result for $\mathbf{E}$ are also valid in the interacting theory. Since Gauss' law contains a spatial derivative, the result for W^0 will look different in the presence of interactions but again the order of magnitude of W^0 is unchanged.

Now consider $k^0 \lesssim g^2T$, ignoring for a moment the physics of $k_0 \sim gT$. To estimate A^0 we need to know the size of $\mathbf{E}(k)$. We have seen that for $k^0 \lesssim g^2T$ the electric field is of the same order of magnitude as ξ. The latter can be estimated from the Fourier

transform of Eq. (4), which gives $\xi(k) \sim k_0^{-1/2}|\mathbf{k}|^{-3/2}$. Then one can use $\nabla A^0(x) \sim \int d^4k e^{-ik\cdot x}\mathbf{E}(k)$, which gives $A^0(x) \sim g^2T$, g^3T for $k^0 \sim g^2T$, g^4T, respectively. Note that these estimates are smaller than the one obtained from the equipartition argument above. This is because the electric field at a given time is dominated by Fourier components with frequencies of order gT. Nevertheless, A_0 is big enough to be able to change the estimate $\partial_0 \sim g^4T$ to $D_0 \sim g^3T$ when D_0 acts on a field with frequency of order g^4T. Still, this would be a factor g smaller than $\mathbf{D} \sim g^2T$, and neglecting D_0 would still be justified.

We are now in the position to tentatively write down an effective theory for the physics associated with $k_0 \lesssim g^2T$ which corresponds to integrating out $\mathbf{E}$ and W^μ as dynamical degrees of freedom. We have seen that for both $k^0 \sim g^2T$ and $k^0 \sim g^4T$ one can neglect the term $D_0\mathbf{E}$. The spatial components of Eq. (1.a) can thus be replaced by

$$\mathbf{D} \times \mathbf{B} = m_{\mathrm{D}}^2 \mathbf{W}. \tag{16.a}$$

In Eq. (1.b) we were able to neglect W^0 both for $k^0 \sim g^2T$ and $k^0 \sim g^4T$. Therefore one can drop W^0 altogether. We have also seen that for $k^0 \sim g^2T$ we could neglect $\mathbf{W}$. For $k^0 \sim g^4T$ we were able to neglect all time derivatives on the lhs of (1.b), and in particular the term $D_0\mathbf{W}$. Thus for both $k^0 \sim g^2T$ and $k^0 \sim g^4T$ the term $D_0\mathbf{W}$ can be neglected and we can replace Eq. (1.b) by

$$3(c_1 + \mathbf{v}\cdot\mathbf{D})\mathbf{v}\cdot\mathbf{W} + (C + v\cdot D)\widetilde{W} = \mathbf{v}\cdot\mathbf{E} + \xi, \tag{16.b}$$

where c_1 is the $l = 1$ eigenvalue of the collision operator C. Eq. (16.a) is no longer a dynamical equation. Instead, it fixes the 3-current in terms of the gauge fields at the same instant of time.

The only remaining question is whether plasmon oscillations affect the low frequency ($k^0 \lesssim g^2T$) dynamics through interactions. To address this issue we consider the gauge field polarization tensor $\Pi^{\mu\nu}(k)$ in the theory (1) at one loop. We are interested in $k_0 \lesssim g^2T$, and loop momenta with $q^0 \sim gT$. Inside the loop one can neglect the effects of C and ξ since $q^0 \gg g^2T$. Without these terms Eq. (1) has the same form as the non-abelian Vlasov equations [5] which describe the hard thermal loop effective theory. Therefore the calculation of $\Pi^{\mu\nu}(k)$ is precisely the same as the one in Ref. [11] where it was found that the leading order in g contribution is due to space-like loop momenta. This shows that the time-like loop momenta $q^0 \sim gT$, $|\mathbf{q}| \sim g^2T$ considered here do not contribute at leading order in g.

Written in $A^0 = 0$ gauge [4], Eq. (16) is a Langevin equation which is purely dissipative, i.e., it contains only first order time derivatives of the dynamical degrees of freedom $\mathbf{A}$ and $\widetilde{W}$,

$$\frac{\partial}{\partial t}\left(\mathbf{v}\cdot\mathbf{A} + \widetilde{W}\right) = -(C + \mathbf{v}\cdot\mathbf{D})\left(3\mathbf{v}\cdot\mathbf{W} + \widetilde{W}\right) + \xi. \tag{17}$$

[4] Note that we have not assumed this gauge when deriving Eq. (16).

For the subsequent discussion I remind the reader of some features of a Langevin equation for some set of degrees of freedom φ_α [25],

$$\frac{\partial}{\partial t}\varphi_\alpha(t) = -f_\alpha[\varphi(t)] + \xi_\alpha(t), \tag{18}$$

where f is a functional of φ at time t. In the present context the index α would represent the spatial coordinates, $\mathbf{v}$ and vector as well as color indices. Both Eq. (6) in $A^0 = 0$ gauge, and Eq. (17) are of this form, as well as the Langevin equation of Ref. [16]. In our case f is a *local* functional of φ, that is, it contains only the φ_α and a finite number of spatial derivatives. In contrast, in the Langevin equation of Ref. [16] f is a spatially non-local functional of the gauge fields [5] . Solving Eq. (18) for any given realization of the noise ξ one generates an ensemble of field configurations for any time t. If the noise is Gaussian and white (that is, it has a frequency independent spectrum),

$$\langle \xi_\alpha(t)\xi_\beta(t')\rangle = 2T\Omega_{\alpha\beta}\delta(t-t'), \tag{19}$$

the probability distribution for field configurations $P(\varphi, t)$ satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial\varphi_\alpha}\left[f_\alpha + T\Omega_{\alpha\beta}\frac{\partial}{\partial\varphi_\beta}\right]P = 0. \tag{20}$$

An important case is that f is related to the derivative of a Hamiltonian H through

$$f_\alpha = \Omega_{\alpha\beta}\frac{\partial H}{\partial\varphi_\beta}. \tag{21}$$

Then the thermal equilibrium distribution $P_{\rm eq} = \exp[-H/T]$ is a stationary solution of the Fokker-Planck equation. Furthermore, for any initial configuration $P(\varphi, t)$ approaches $P_{\rm eq}$ for large times.

For Eq. (6) we have $\Omega_{\alpha\beta} = \gamma^{-1}\delta_{\alpha\beta}$ and $f_\alpha = \Omega_{\alpha\beta}(\partial H_{\rm 3d}/\partial\varphi_\beta)$ with the Hamiltonian

$$H_{\rm 3d} = \frac{1}{2}\int d^3x\mathbf{B}^2. \tag{22}$$

Thus for Eq. (6) the probability distribution approaches $\exp[-H_{\rm 3d}/T]$ for large times. Note that $H_{\rm 3d}/T$ is the action of magnetostatic Yang-Mills theory [26] which is obtained for equilibrium quantities by dimensional reduction and by integrating out the A_0 field.

Beyond the leading and next-to-leading log approximation, for which Eq. (1) reduces to Eq. (6), it has so far not been understood whether Eq. (1) reproduces the correct thermodynamics of long distance Yang-Mills fields. The noise correlator of Ref. [16] depends on the gauge fields, and is therefore not of the form (19). With a field dependent noise correlator the time discretisation of the Langevin equation is ambiguous.

[5] In order to avoid confusion I would like to stress the following. By reintroducing the W-field the Langevin equation of Ref. [16] can be written in a local form. Then, however, the equation for W would be a *constraint* and not an equation of motion.

In Ref. [16] this ambiguity was fixed by *postulating* that the Langevin equation yields the equilibrium distribution $\exp[-H_{\rm 3d}/T]$. The considerations of Ref. [27] did not take into account the non-linear character of Eq. (1.b).

We will now determine the equilibrium distribution which is generated by Eq. (17). Obviously the rhs of Eq. (17) can not be written like in Eq. (18) with an f of the form (21). It is well known, however, that the equilibrium distribution does not specify the function f_α uniquely [25]. One may add an extra term F_α,

$$f_\alpha = \Omega_{\alpha\beta}\frac{\partial H}{\partial\varphi_\beta} + F_\alpha, \tag{23}$$

without changing the equilibrium distribution provided that F_α satisfies

$$\frac{\partial F_\alpha}{\partial\varphi_\alpha} = F_\alpha\frac{\partial H}{\partial\varphi_\alpha}. \tag{24}$$

We will see that our Langevin equation (17) is indeed of the form (18), (23) with an F satisfying the condition (24). From Eq. (4) we see that we have to identify

$$\Omega_{\alpha\beta} \leftrightarrow \frac{1}{m_{\rm D}^2}C(\mathbf{v},\mathbf{v}')\delta(\mathbf{x}-\mathbf{x}')\delta^{ab}. \tag{25}$$

The next question concerns the relevant Hamiltonian. Eq. (1) was obtained from the Hard Thermal Loop effective theory for which the Hamiltonian reads [28]

$$H_{\rm HTL} \;=\; \frac{1}{2}\int d^3x\left\{\mathbf{E}^2+\mathbf{B}^2+m_{\rm D}^2\int_{\mathbf{v}}W^2\right\}. \tag{26}$$

The effective theory (16) is obtained from Eq. (1) by integrating out $\mathbf{E}$ and W^μ. Thus one can expect that the relevant Hamiltonian is obtained from (26) by dropping these fields, i.e.,

$$H \;=\; \frac{1}{2}\int d^3x\left\{\mathbf{B}^2+m_{\rm D}^2\int_{\mathbf{v}}\widetilde{W}^2\right\}. \tag{27}$$

Using

$$\frac{\delta H}{\delta\mathbf{A}} = \mathbf{D}\times\mathbf{B}, \quad \frac{\delta H}{\delta\widetilde{W}} = m_{\rm D}^2\widetilde{W}, \tag{28}$$

together with Eq. (25) we indeed obtain the terms on the rhs of Eq. (17) which contain the collision term. The remaining terms on the rhs of Eq. (17) have to be identified with F_α,

$$F_\alpha \leftrightarrow -\mathbf{v}\cdot\mathbf{D}\left(3\mathbf{v}\cdot\mathbf{W}+\widetilde{W}\right). \tag{29}$$

We will now see that for this F_α both the lhs and the rhs of Eq. (24) are zero, so that Eq. (24) is indeed satisfied. First consider the lhs of Eq. (24). When the φ-derivative acts on the gauge field contained in $\mathbf{v}\cdot\mathbf{D}$ one obtains zero due to the contraction of color

indices and the antisymmetry of the structure constants. When it acts on $3\mathbf{v}\cdot\mathbf{W}+\widetilde{W}$ the result is an integral over a total spatial derivative which again vanishes. The rhs of Eq. (24) can be written as

$$-\frac{1}{m_{\rm D}^4}\int d^3x\int_{\mathbf{v}}\frac{\delta H}{\delta\varphi(x,\mathbf{v})}\mathbf{v}\cdot\mathbf{D}\frac{\delta H}{\delta\varphi(x,\mathbf{v})},$$

with $\varphi\equiv\mathbf{v}\cdot\mathbf{A}+\widetilde{W}$. This again is an integral of a total derivative and vanishes.

We conclude that the probability distribution generated by the Langevin equation (17), with $\mathbf{W}$ given by Eq. (16.a), approaches for large times the Boltzmann distribution $P_{\rm eq}=\exp(-H/T)$ with the Hamiltonian (27). This is not the equilibrium distribution corresponding to dimensional reduction because it contains the additional field $\widetilde{W}$. In H, however, $\widetilde{W}$ is not coupled to the gauge fields. Equal time correlation functions of $\mathbf{A}$ computed with the help of H are thus the same as in 3-dimensional Yang-Mills theory.

Another interesting observation is the following. Without a kinetic term $D_0\mathbf{E}$ there are no propagating gauge field waves in the theory (16). These would cause the same non-local UV divergences as in classical thermal Yang-Mills theory [20], which are still present in Eq. (1) since the effect of the W fields can be neglected in the ultraviolet. Therefore the UV divergences of Eq. (16) can be expected to be local.

To summarize, we have found that Eq. (1) describes plasmon oscillations, multipole fluctuations of color charge, and the non-perturbative gauge field dynamics. These processes are associated with characteristic frequencies gT, g^2T, and g^4T, respectively. An effective theory (16) was constructed which reproduces the slow ($k_0\lesssim g^2T$) dynamics at leading order in g. Previously [6, 22, 16, 12] it was implicitly assumed that all modes with $k_0\gg g^4T$ decouple from the non-perturbative gauge field dynamics. Here it was shown that this is indeed the case for the plasmon oscillation. To see whether the same is true for the multipole fluctuations requires a more detailed analysis.

Acknowledgements I would like to thank P. H. Damgaard, M. Laine, G.D. Moore and K. Rummukainen for useful discussions and suggestions. This work was supported by the TMR network "Finite temperature phase transitions in particle physics", EU contract ERBFMRXCT97-0122.

References

[1] A. D. Linde, Phys. Lett. **B96** (1980) 289; D. J. Gross, R. D. Pisarski, L. G. Yaffe, Rev. Mod. Phys. **53** (1981) 43.

[2] For reviews of high temperature dimensional reduction see, e.g., M. E. Shaposhnikov, hep-ph/9610247; A. Nieto, Int. J. Mod. Phys. **A12**, 1431 (1997) [hep-ph/9612291]; M. Laine, hep-ph/9707415.

[3] For a review of electroweak baryon number violation and baryogenesis see, e.g., V.A. Rubakov, M.E. Shaposhnikov, Usp. Fis. Nauk **166** (1996) 493 [hep-ph/9603208].

[4] E. Braaten and R. Pisarski, Nucl. Phys. **B337** (1990) 569; Phys. Rev. **D45** (1992) 1827; J. Frenkel and J.C. Taylor, Nucl. Phys. **B 334** (1990) 199; J.C. Taylor and S.M.H. Wong, Nucl. Phys. **B346** (1990) 115.

[5] J. P. Blaizot, E. Iancu, Nucl. Phys. **B417**, 608 (1994) [hep-ph/9306294]; V.P. Nair, Phys. Rev. **D48**, 3432 (1993) [hep-ph/9307326].

[6] D. Bödeker, Phys. Lett. **B426**, 351 (1998) [hep-ph/9801430].

[7] P. Arnold, D. T. Son, L. G. Yaffe, Phys. Rev. **D59**, 105020 (1999) [hep-ph/9810216].

[8] M. A. Valle Basagoiti, hep-ph/9903462;

[9] D. F. Litim and C. Manuel, Phys. Rev. Lett. **82**, 4981 (1999) [hep-ph/9902430];

[10] J. Blaizot and E. Iancu, Nucl. Phys. **B557**, 183 (1999) [hep-ph/9903389].

[11] D. Bödeker, Nucl. Phys. **B566**, 402 (2000) [hep-ph/9903478].

[12] P. Arnold, L. G. Yaffe, Phys. Rev. **D62**, 125014 (2000) [hep-ph/9912306]; Phys. Rev. **D62**, 125013 (2000) [hep-ph/9912305].

[13] The lhs of Eq. (6) can be estimated as $(g^2T)^2\mathbf{A}$, while on the rhs $\gamma\mathbf{E} \sim T/\log(1/g)k_0\mathbf{A}$. This gives $k_0 \sim \log(1/g)g^4T$.

[14] G. D. Moore, Nucl. Phys. **B568**, 367 (2000) [hep-ph/9810313].

[15] G.D. Moore, Phys. Rev. **D62**, 085011 (2000) [hep-ph/0001216].

[16] P. Arnold, Phys. Rev. **D62**, 036003 (2000) [hep-ph/9912307].

[17] G.D. Moore, C. Hu, B. Müller, Phys. Rev. **D58**, 045001 (1998) [hep-ph/9710436].

[18] G. D. Moore, K. Rummukainen, Phys. Rev. **D61**, 105008 (2000) [hep-ph/9906259].

[19] D. Bödeker, G.D. Moore, K. Rummukainen, Phys. Rev. **D61**, 056003 (2000) [hep-ph/9907545].

[20] D. Bödeker, L. McLerran, A. Smilga, Phys. Rev. **D52**, 4675 (1995) [hep-th/9504123].

[21] P. Arnold, D. Son, L. G. Yaffe, Phys. Rev. **D55**, 6264 (1997) [hep-ph/9609481]; P. Huet, D. T. Son, Phys. Lett. **B393**, 94 (1997) [hep-ph/9610259].

[22] D. Bödeker, Nucl. Phys. **B559**, 502 (1999) [hep-ph/9905239].

[23] P. Arnold, D.T. Son, L.G. Yaffe, Phys. Rev. **D60**, 025007 (1999) [hep-ph/9901304].

[24] E.M. Lifshitz, L.P. Pitaevskii, *Physical Kinetics* (Pergamon Press, Oxford 1981).

[25] For an introduction and review, see J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press, Oxford, 1993.

[26] E. Braaten and A. Nieto, Phys. Rev. D **53** (1996) 3421 [hep-ph/9510408].

[27] D. F. Litim and C. Manuel, Phys. Rev. D **61**, 125004 (2000) [hep-ph/9910348].

[28] J. Blaizot and E. Iancu, Nucl. Phys. **B421**, 565 (1994) [hep-ph/9401211]; V.P. Nair, Phys. Rev. **D50**, 4201 (1994) [hep-th/9403146].